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Ewald-Oseen Extinction Theorem

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1 Introduction

The Ewald-Oseen extinction theorem provides an insight into how materials respond when light is shone onto them. When I say "material", think of substances like water or glass for which you can easily imagine light going through. The theorem also applies to opaque materials but transparent stuff make better examples. So, when you take your flashlight and shine light onto the pond in your backyard, what happens to the light and the water? How does it happen? These are the questions that the theorem answers.

Light is an electromagnetic wave. Some people reserve the word "light" specifically for the "visible light" that your eyes can see, excluding, for example, the Wi-Fi signals your mobile phone sends and receives, or the x-rays that go through your body when you have a cat scan at the hospital. Other people use "light" more flexibly, including infrared (lower frequency) and ultraviolet (higher frequency) waves too. For my purposes here, electromagnetic wave and light mean the same thing and I will use them interchangeably.

The theorem falls into the branch of physics called Electrodynamics which explains, in the form of four equations, how the light behaves at the fundamental level. These equations are called Maxwell's equations. It is my job to solve and use the results of those equations to draw you an intuitive picture. Historically, Paul Peter Ewald (in 1912) and Carl Wilhelm Oseen (in 1915) were the first people to conceive of the ideas behind this theory, hence the name. The theory has come to be what it is today only after several corrections and modifications along the years, which make it difficult to track the current version. This is perhaps the sole reason why it is still widely misunderstood, even by experienced scientists. This is also one of the reasons why I decided to write my take on the theorem, trying to make it comprehensible by a broad range of readers from electromagnetics veterans to curious laymen. With an aim to satisfy the readers from the opposite ends of this spectrum, I present the topic in a popular science fashion but also give the mathematical derivations in the appendices.

The theorem adopts a microscopic point of view and derives the manifestations that one observes in macroscopic world (I will shortly explain what exactly I mean by microscopic and macroscopic). The way the theorem visualizes the reflection and transmission of an electromagnetic wave is enlightening, on which I hope

you will agree with me by the time you finish reading this paper. I believe that a thorough understanding of the physical phenomena is a key characteristic of an accomplished scientist, regardless of the subject of study. This paper aims to advance this understanding by explaining a usually overlooked and seemingly complicated topic in an appealing manner.

2 Remind me, what happens when light enters into water?

Look at Figure 1. It shows a ray of light entering the water in the glass. The light refracts (bends) as it crosses the boundary between air and water. This is why the straw in a glass of your favorite drink (mine is water; yes, I am that boring) looks broken, a phenomenon I am sure many of you observed. Note that some of the light also reflects off the water's surface, which I am currently not interested in.

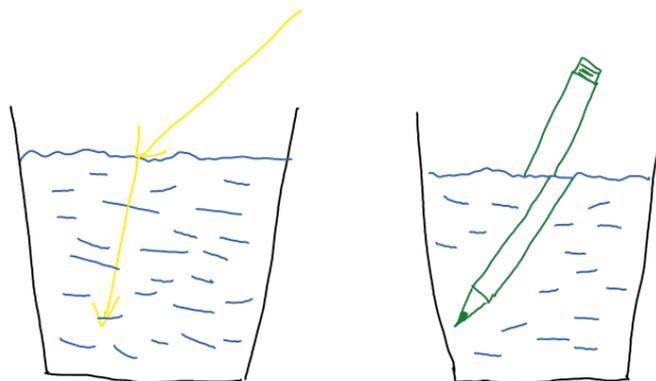


Figure 1: Light rays in water

The light also slows down in water. At least, this is the common understanding within the scientific community. These two properties of light (refraction and slowing down) are very intuitive: First, you can definitely observe refraction as in Figure 1, and second, you can imagine almost everything slowing down when they enter the water. For example, a bullet shot into the water would slow down tremendously and come to a full stop after a few meters. Even you would slow down in water, as it is harder to move in water than in air. The reason is clear

and sound: The water is *denser* than air. Therefore, it makes sense to expect the light to behave similarly and slow down in water.

I have established two properties of light: It slows down and changes its direction when it enters a denser medium. All I did was make an observation and use an analogy. This is, of course, far from convincing, so I will talk about how these are proven mathematically in the next section.

3 Proving how lights slows down and changes direction

One uses Maxwell's equations and boundary conditions to prove these properties mathematically. As I said, Maxwell's equations are simply the rules that electromagnetic waves obey, just like planets obey the rules of gravitation. Then, what are boundary conditions? They are the general results scientists derive from Maxwell's equations, so that they can use them when they want to solve a specific problem.

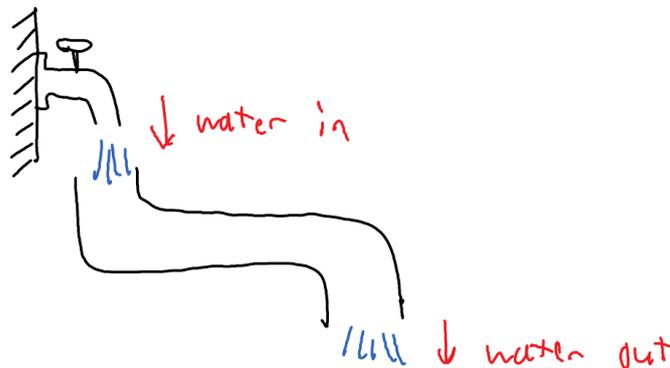


Figure 2: Boundary conditions

Look at Figure 2. A tap pumps water into a water pipe. The water comes out at the other end of the pipe. The boundary is defined as the input and output openings of the pipe. However much water is pumped, it will all exit through the output opening (unless there is a crack in the pipe). This rule is what is called the boundary condition: *What goes in must come out!* The shape or the windings of the pipe do not matter. If I know where the input and output openings are, then

I can easily calculate the output for any given input. In electromagnetics, a very similar boundary condition is used. The only difference is that it is light which flows, not water.

Boundary conditions make my life a lot easier. The calculations are much simpler, but now I have a different problem. I do not know what the heck is going on in the water pipe! This is exactly where the Ewald-Oseen extinction theorem comes in and explains what is happening *microscopically* (things that are small and cannot be observed easily, i.e. inside the pipe) in addition to the macroscopic results that I can already calculate (things that are big and easily observable, i.e. the amount of water that comes out and where it comes out). By opting for boundary conditions, I traded the information about the pipe's interior with computational ease.

As I said, the conventional approach for solving an electromagnetic problem is using boundary conditions. Other approaches, including the microscopic point of view, usually prove to be too difficult. I will investigate the simplest case in this paper. However, I will be able to derive the most important conclusions of the Ewald-Oseen extinction theorem.

You might very well ask why I even care about what is happening inside the pipe. I care because I am curious. I have an inquisitive mind, and I believe so do you! In the end, it pays to have questioned every single detail. You might find errors in your current understanding of the nature of things, as you will in this case. I will show that the two properties of light that I found previously (slowing down and refracting/bending) are indeed wrong. They are illusions caused by the oversimplification I did by defining the boundary conditions.

4 What is this Ewald-Oseen extinction theorem about?

Now, I define the situation in a more scientific (and less fun) way. Figure 3 is the same as the water in the glass in Figure 1, except that I assume half of the universe is filled with a dielectric material (upper region), the other half is vacuum (lower region) in Figure 3. I use the word *dielectric* to refer to the class of materials such as water, glass, oil, but not metals such as copper and gold.

The electromagnetic plane wave initially travels in the semi-infinite vacuum, and then hits the boundary of the semi-infinite dielectric medium at $z=0$. (*Plane wave* is a fancy way of saying that the wave travels in a single direction and has only one frequency/color. There are other types of waves which are a mixture of several frequencies and travel in many directions.) Part of the incident wave *reflects* back into the vacuum (indicated with k_r), and the remainder is *transmitted* into the dielectric medium (indicated with k_t). The k 's are vectors indicating the directions of the waves. We, scientists, use ϵ and μ parameters instead of words like "water" or "air" to differentiate between different materials ($\epsilon_{water} = 1.77$, and $\epsilon_{vacuum} = 1$, which is why we call it ϵ_0 . μ 's are assumed to be 1 for simplicity). I will use those parameters later in my calculations.

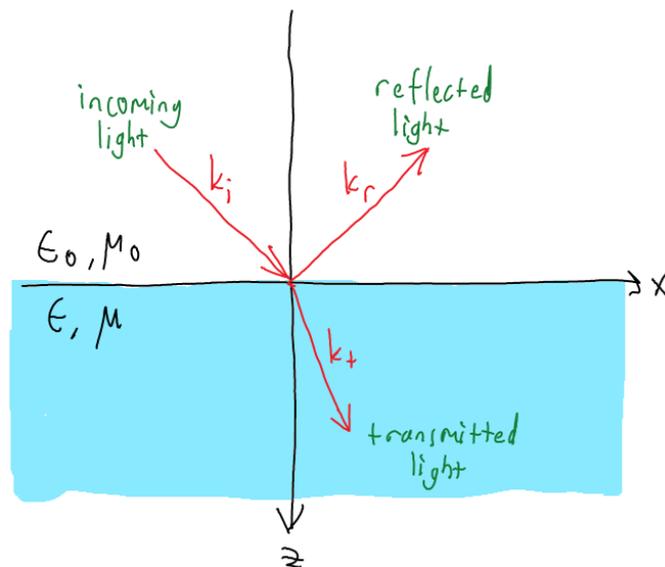


Figure 3: Electromagnetic wave incident on a semi-infinite dielectric medium - Macroscopic point of view

Figure 3 depicts one of the simplest electromagnetic phenomena; I can use Maxwell's boundary conditions to find the reflected and the transmitted waves. I will not do that, but the conclusions I would arrive at are the following.

- The reflection occurs at the same angle as the angle of incidence.
- Transmitted wave travels at a different angle and at a slower speed.

- Both the transmitted and the reflected waves are still plane waves.

I believe that these conclusions are remarkable. Let me explain why. A dielectric is a collection of atoms (or molecules), and rest is simply empty space as shown in Figure 4. Atoms occupy only a tiny fraction of the material, so most of it is vacuum. Figure 4 is, if you prefer, the interior of the water pipe. All the incident wave does is excite those atoms. The behaviour of an individual atom under excitation is much more complex than a plane wave. An atom radiates electromagnetic waves in all directions (the exact derivation is given in Appendix A). If I did not know any better, I would expect a chaos of electromagnetic waves.

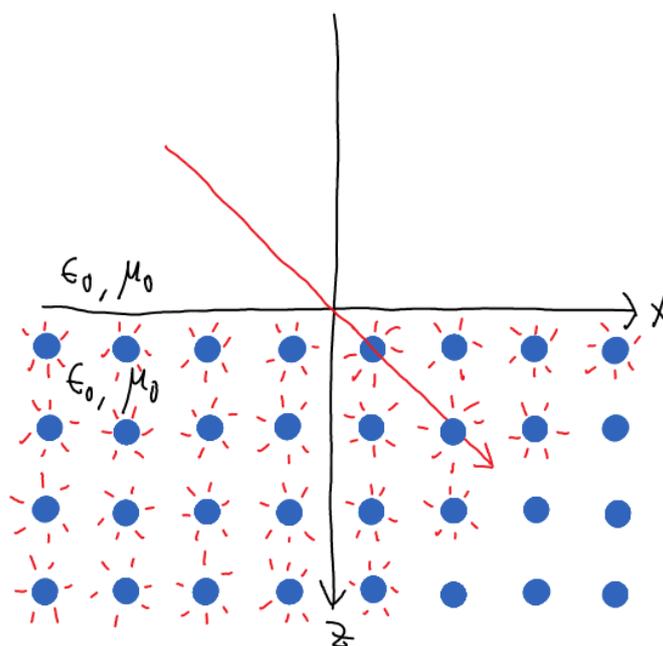


Figure 4: Ewald-Oseen extinction theorem - Microscopic point of view (Blue: Atoms, Red: Electromagnetic radiation)

First, the incident wave travels at the speed of light in vacuum, c , which is about 300,000 kilometers per second. The speed of light in vacuum is a universal constant. The waves created by the atoms also travel at the speed of light c because they basically travel in vacuum. Then, how does the wave actually slow down?

Second, the incident wave travels in a single direction. The atoms radiate waves in all directions. So, how do I observe the light in water as if it is going in a single

and different direction?

Do these atoms or molecules conspire to give us these unexpected results, or is there something else at play? The conventional way of solving this problem (using boundary conditions) does not answer any of these questions because it does not really explain how the dielectric behaves at the microscopic level. It assumes that the medium has a macroscopic description and follows from there. I will answer all of these questions using the Ewald-Oseen extinction theorem. I will calculate the fields generated by the atoms, sum them up with my incident field to find the resultant electromagnetic waves, and also answer the questions in the meantime.

5 Mistaking simplifications with the reality

In the previous sections, I have provided two contradictory expectations about the behaviour of light! First, I gave examples of things that slow down in water or look broken when partially in water, and said the logical guess would be that the light behaved the same way. Afterwards, I drew a microscopic picture with atoms and claimed that there is no reason for the light to slow down because everywhere is basically vacuum. In this section, I would like to talk about the danger I put myself in when I use my equations and try to apply my everyday experiences to a phenomenon which I cannot directly observe.

Take the boundary conditions for instance. In Figure 5, I give a mundane example. I get a ball and throw it against the wall. It travels in air mostly undisturbed. It hits the wall and it bounces (reflects) back to me. It also transmits some of its energy to the wall as heat or vibration due to collision. In this picture, it is easy to identify the role of the boundary between the wall and the air. It is where the collision, thus the energy transfer takes place. If there were not a boundary, then there would not be a collision and the ball would not bounce at all.

At first sight, all of these arguments seem to apply to light as well. After all, I can define the electromagnetic boundary conditions and use them to solve real problems and verify my observations. But, here is the catch: A closer look at what I call a material reveals that it is merely a collection of atoms and therefore the definition of the boundary becomes vague (Figure 4). Where exactly is the boundary? If it is the first layer of atoms, what about the next layers of atoms

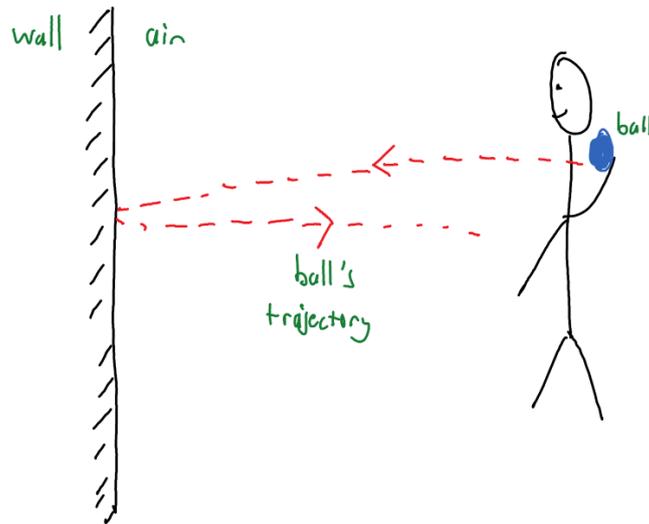


Figure 5: Throwing a ball against the wall - The role of the boundary

which will surely interact with the incident light? What is reflection, if the incident light just goes through the atoms which, in return, generate electromagnetic waves that go in all directions? All of these concepts such as material, boundary and reflection lose their meanings at the microscopic level.

What I should deduce from this discussion is that I must be careful when I try to interpret the nature (how things really are) using my equations and past experiences. The electromagnetic boundary conditions are just *mathematical tools* that help me get to the end result. They do not show the physical picture behind the scenes. I agree with Richard Feynman. Nature is what it is, not what I expect it to be, or not what I think is logical. Meaningless as it seems to ask how much of the ball reflects back or if it goes through the wall, it is wrong *not* to ask such questions when encountered with new physical phenomena. This theorem is the fruit of asking such questions.

Section 6 is where I derive the entire theorem. I will arrive at the following conclusions:

- The incident light neither reflects nor refracts when it enters a new medium. It does not slow down either. All it does is travel in a single direction *everywhere* and excite the atoms along the way.

- In response, the excited atoms produce waves which also travel at the speed of light in vacuum. When all the waves from the atoms are added up in the lower half-space (see Figure 4), a portion of those waves exactly cancel the incident field. What is left is a wave that *seems* to travel in a single direction at a lower speed. In reality, it is a combination of many waves which travel with speed c . In the upper half-space, again, the summation of all the waves from the atoms look like a single wave which travels in a single direction (observed as the reflected wave).

I highly encourage you to read the next section and witness how the conclusions above come about with your own eyes. However, if you do not want to deal with any math, this paragraph marks the end of this paper for you.

Those who are not convinced yet, I present you the thorough proof of the theorem. I would not be convinced either. I would demand strong evidence that backs such outrageous claims. The next section follows the same scenario: An initial wave excites the atoms approximated as electric dipoles, and then the waves of the dipoles are added. Although very easy to state in words, a mathematical proof is a little bit more involved. It is my aim to present it as clearly as possible.

6 Derivation of Ewald-Oseen extinction theorem

6.1 Setting up the problem

I will assume a normally incident plane wave with time dependence $e^{-i\omega t}$. The incident electric field is

$$\mathbf{E}_i(z, t) = \hat{\mathbf{x}}E_i(z)e^{-i\omega t} = \hat{\mathbf{x}}E_{i0}e^{ikz-i\omega t} \quad (1)$$

I also define the total field and the field due to the dipoles as

$$\mathbf{E}_t(z, t) = \hat{\mathbf{x}}E_t(z)e^{-i\omega t} \quad (2)$$

$$\mathbf{E}_d(z, t) = \hat{\mathbf{x}}E_d(z)e^{-i\omega t} \quad (3)$$

where t stands for total and d stands for dipole. Then, the total field at any given point \mathbf{r} is the sum of incident and dipole fields (superposition principle).

$$\mathbf{E}_t(z, t) = \mathbf{E}_i(z, t) + \mathbf{E}_d(z, t) \quad (4)$$

Here is the Maxwell's equations to refer to in the following sections.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (5)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (6)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (7)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (8)$$

or in macroscopic media

$$\nabla \cdot \mathbf{D} = \rho_{ext} \quad (9)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (10)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (11)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_{ext} + \frac{\partial \mathbf{D}}{\partial t} \quad (12)$$

where $_{ext}$ stands for external.

6.2 Fields of a single dipole

I would like to find $\mathbf{E}_d(z, t)$, so that I can calculate $\mathbf{E}_t(z, t)$ and see if it matches my expectations. First, I need to find the fields generated by a single dipole. I will take it from the top and start with Maxwell's equations for completeness. Since $\nabla \cdot \mathbf{B} = 0$ and $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ for any \mathbf{A} , I can define a *vector potential* \mathbf{A} such that $\nabla \times \mathbf{A} = \mathbf{B}$. Then,

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mathbf{A} \longrightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \quad (13)$$

Similarly, I can define a scalar potential Φ such that

$$-\nabla \Phi = \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \longrightarrow \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi \quad (14)$$

Using equation (5) (Gauss' law),

$$\nabla^2 \Phi + \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -\frac{\rho}{\epsilon_0} \quad (15)$$

Using equation (8) (Ampere's law),

$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left(\nabla \cdot \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial \Phi}{\partial t} \right) = -\mu_0 \mathbf{J} \quad (16)$$

I have only defined the curl of the vector potential \mathbf{A} . This means I am free to choose the divergence however I want and I choose it as follows.

$$\nabla \cdot \mathbf{A} = \mu_0 \epsilon_0 \frac{\partial \Phi}{\partial t} \quad (17)$$

This choice is called the Lorenz gauge. Another widely known choice is the Coulomb gauge, which I will not discuss. Using this gauge in equations (15) and (16),

$$\nabla^2 \Phi - \mu_0 \epsilon_0 \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \quad (18)$$

$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \quad (19)$$

The solution for the vector potential \mathbf{A} is given by

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{R} d^3 r' = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \frac{e^{-i\omega t_r}}{R} d^3 r' = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \frac{e^{ikR - i\omega t}}{R} d^3 r' \quad (20)$$

where $t_r = t - R/c$ is the retarded time, \mathbf{r} is the observation point, \mathbf{r}' is the location of the dipole, and $R = |\mathbf{r} - \mathbf{r}'|$, as seen in Figure 6. The solution for Φ is the same as \mathbf{A} , except that $\mu_0 \mathbf{J}$ is replaced by ρ/ϵ_0 . If you are curious about how I obtained the solution to these differential equations, see Appendix D.

I am considering ideal dipoles which are infinitesimally small in size; therefore, we can take the exponential and $1/r$ term out of the integral in equation (20).

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{e^{ikR}}{R} \int \mathbf{J}(\mathbf{r}') d^3 r' \quad (21)$$

I will evaluate this integral with the help of the continuity equation.

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = i\omega \rho \quad (22)$$

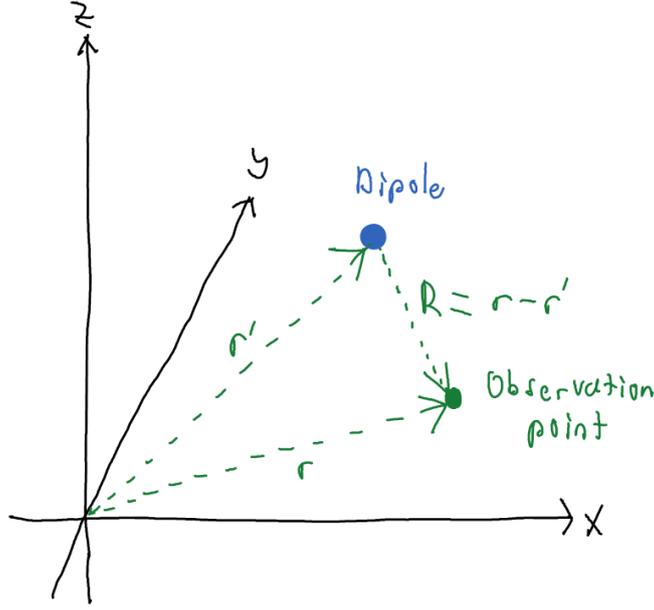


Figure 6: A single dipole in free space.

Apply integration by parts to the integral.

$$\int \mathbf{J}(\mathbf{r}') d^3 r' = \mathbf{r} \cdot \mathbf{J}|_{r=0} - \int \mathbf{r}' \nabla' \cdot \mathbf{J}(\mathbf{r}') d^3 r' \quad (23)$$

where the first term on the right-hand side is zero. Substitute equation (22) and (23) into (21).

$$\mathbf{A}(\mathbf{r}) = -\frac{i\omega\mu_0}{4\pi} \frac{e^{ikR}}{R} \int \mathbf{r}' \rho(\mathbf{r}') d^3 r' = -\frac{i\omega\mu_0}{4\pi} \mathbf{p} \frac{e^{ikR}}{R} \quad (24)$$

where

$$\mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}') d^3 r' \quad (25)$$

is the electrical dipole moment. Now, I use the Lorenz gauge conditions to find the fields.

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (26)$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad (27)$$

Evaluating equation (26) gives

$$\mathbf{H} = \frac{1}{\mu_0} \nabla \times \mathbf{A} = \frac{\omega}{4\pi} e^{ikR} \left(\frac{k}{R} + \frac{i}{R^2} \right) \hat{\mathbf{R}} \times \mathbf{p} \quad (28)$$

Putting \mathbf{H} into the last Maxwell's equation (8),

$$\begin{aligned} \mathbf{E} &= \frac{i}{\omega\epsilon_0} \nabla \times \mathbf{H} \\ &= \frac{e^{ikR}}{4\pi\epsilon_0} \left[\left(3\hat{\mathbf{R}}(\hat{\mathbf{R}} \cdot \mathbf{p}) - \mathbf{p} \right) \left(\frac{1}{R^3} - \frac{ik}{R^2} \right) - \left(\hat{\mathbf{R}}(\hat{\mathbf{R}} \cdot \mathbf{p}) - \mathbf{p} \right) \frac{k^2}{R} \right] \end{aligned} \quad (29)$$

Deriving equations (28) and (29) is indeed cumbersome, but nothing more than a vector algebra exercise. The complete derivation is given in Appendix A for interested readers.

6.3 Summation of all dipoles

I have found the fields of a single dipole, so what is left is to find $\mathbf{E}_d(\mathbf{r}, t)$ by summing the fields of all dipoles. A dipole will be polarized due to the total electric field acting on it. In this case, the total field acting on a dipole is the incident electric field plus the fields generated by all the other dipoles! This is, indeed, the same as the total electric field I defined previously in equation (2), namely $\mathbf{E}_t(z, t) = \hat{\mathbf{x}}E_t(z)e^{-i\omega t}$. Moreover, the direction of the polarization will be in the direction of the total electric field. So,

$$\mathbf{p} \propto E_t(z)\hat{\mathbf{x}} \quad (30)$$

If I assume that the dipoles form a continuous distribution in $z > 0$ region, the dipole moment of a unit volume will simply be the polarization density of the medium.

$$\mathbf{p} = (\epsilon - \epsilon_0)E_t(z)e^{-i\omega t}\hat{\mathbf{x}} \quad (31)$$

This is justified because there are already many dipoles in the smallest volume that I can possibly be interested in. Now, I have everything to construct the integral over \mathbf{E} in equation (29) in $z > 0$ region. Putting \mathbf{p} into (29) and multiplying by $\hat{\mathbf{x}}$ to get the amplitude,

$$E_d(z) = \frac{\epsilon_r - 1}{4\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_0^{\infty} dz' \\ \times E_t(z') e^{ik_0 R} \left[\left(3(\hat{\mathbf{R}} \cdot \hat{\mathbf{x}})^2 - 1 \right) \left(\frac{1}{R^3} - \frac{ik_0}{R^2} \right) - \left(3(\hat{\mathbf{R}} \cdot \hat{\mathbf{x}})^2 - 1 \right) \frac{k_0^2}{R} \right] \quad (32)$$

where ϵ_r is the relative permittivity, and k is replaced with k_0 to emphasize that it is the free space wave vector.

Evaluating the integrals over x' and y' is cumbersome. I will simply give the result here. Curious readers can check Appendix B.

$$E_d(z) = \frac{1}{2} ik_0 (\epsilon_r - 1) \int_0^{\infty} dz' E_t(z') e^{ik_0 |z - z'|} \quad (33)$$

Let me explain this integral. The only discontinuity in the space occurs in z direction, so the results must be invariant with respect to x and y directions. Integrating over x and y gives us a slice of dipoles in the z direction. This slice radiates plane waves (due to the polarizing field acting on it) in both positive and negative z directions. Each slice radiates a plane wave with a different magnitude and phase, and all this integral does is add these plane waves coherently. That's it! Actually, a different approach to prove Ewald-Oseen extinction theorem starts with this observation [2].

Finally, I can write the total field as

$$E_t(z) = E_{i0} e^{ik_0 z} + \frac{1}{2} ik_0 (\epsilon_r - 1) \int_0^{\infty} dz' E_t(z') e^{ik_0 |z - z'|} \quad (34)$$

6.4 Electric field inside the dielectric

I start by dividing the integral in (34) into two parts.

$$E_t(z) = E_{i0} e^{ik_0 z} + \frac{1}{2} ik_0 (\epsilon_r - 1) \\ \times \left[e^{ik_0 z} \int_0^z dz' E_t(z') e^{-ik_0 z'} + e^{-ik_0 z} \int_z^{\infty} dz' E_t(z') e^{ik_0 z'} \right] \quad (35)$$

A visual of this integral is shown in Figure 7. The first integral from 0 to z represent forward propagating waves and originate from the dipole layers below z . The second integral represent backward propagating waves and originate from

layers above z . This is exactly the same observation we made in the previous section. Notice that E_t is a function of its integral. To make sense of this, I visualize the following series of events: First, a single dipole is excited by the summation of the incident field and the fields due to all other dipoles. This dipole, in return, generates fields which affect all other dipoles. Then, all the other dipoles adjust their fields accordingly and act upon the first dipole, and so on. This is, of course, a transient picture of what is happening. Digging too much into it may lead to wrong impressions. After all, scientists almost always observe the steady-state result.

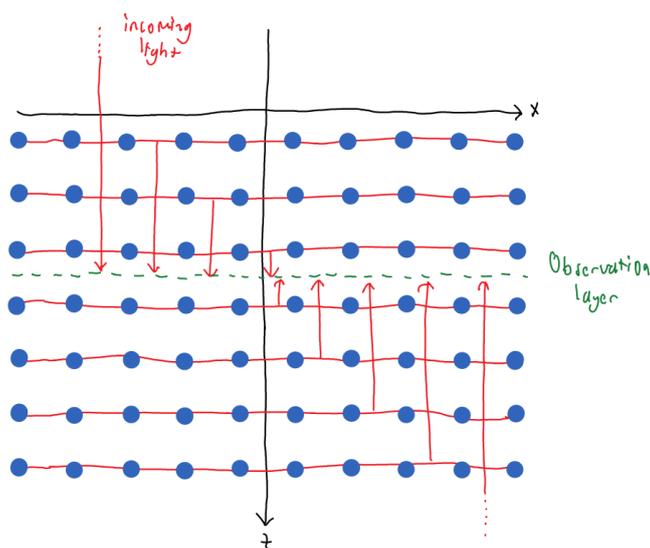


Figure 7: Forward and backward propagating waves originating from different layers and contributing the total field at the observation layer

I use a trial solution to solve the equation for $z > 0$,

$$E_t(z) = E_{t0}e^{ikz} \quad (36)$$

where k and E_{t0} are unknown.

$$E_t(z) = E_{i0}e^{ik_0z} + \frac{1}{2}ik_0(\epsilon_r - 1) \times \left[e^{ik_0z} E_{t0} \int_0^z dz' e^{i(k-k_0)z'} + e^{-ik_0z} E_{t0} \int_z^\infty dz' e^{i(k+k_0)z'} \right] \quad (37)$$

$$E_t(z) = E_{i0} e^{ik_0 z} + \frac{1}{2} i k_0 (\epsilon_r - 1) E_{t0} \times \left[\frac{e^{ik_0 z}}{i(k - k_0)} (e^{i(k-k_0)z} - 1) - \frac{e^{-ik_0 z}}{i(k + k_0)} e^{i(k+k_0)z} \right] \quad (38)$$

$$E_t(z) = e^{ik_0 z} \left[E_{i0} - \frac{1}{2} \frac{k_0 (\epsilon_r - 1)}{k - k_0} E_{t0} \right] + e^{ikz} \frac{k_0^2 (\epsilon_r - 1)}{k^2 - k_0^2} E_{t0} \quad (39)$$

Note that while evaluating the integrals I used

$$\lim_{z \rightarrow +\infty} e^{ikz} = 0 \quad (40)$$

This is mathematically incorrect, but I use my physical intuition to see that the waves will sooner or later decay and won't be able to reach infinity.

For equation (39) to hold, I need the following two conditions.

$$\frac{k_0^2 (\epsilon_r - 1)}{k^2 - k_0^2} = 1 \quad (41)$$

and

$$E_{i0} - \frac{1}{2} \frac{k_0 (\epsilon_r - 1)}{k - k_0} E_{t0} = 0 \quad (42)$$

From condition (41), I get $k^2 = k_0^2 \epsilon_r \rightarrow k = k_0 \sqrt{\epsilon_r} = k_0 n$. From condition (42), we get

$$E_{t0} = E_{i0} \frac{2k_0(n-1)}{k_0(n^2-1)} = E_{i0} \frac{2}{n+1} \quad (43)$$

These are the well-known results of the conventional (macroscopic) approach.

Now, I calculate the field generated by the dipoles. Putting E_t into equation (33),

$$\begin{aligned} E_d(z) &= \frac{1}{2} i k_0 (\epsilon_r - 1) \int_0^\infty dz' e^{ik_0|z-z'|} e^{ikz'} \frac{2}{n+1} E_{i0} \\ &= i k_0 (n-1) E_{i0} \left[e^{ik_0 z} \int_0^z dz' e^{i(k-k_0)z'} + e^{-ik_0 z} \int_z^\infty dz' e^{i(k+k_0)z'} \right] \\ &= i k_0 (n-1) E_{i0} \left[\frac{e^{ik_0 z}}{i(k-k_0)} (e^{i(k-k_0)z} - 1) - \frac{e^{-ik_0 z}}{i(k+k_0)} e^{i(k+k_0)z} \right] \\ &= -E_{i0} e^{ik_0 z} + \frac{2}{n+1} E_{i0} e^{ik_0 n z} \end{aligned} \quad (44)$$

I see that it exactly cancels the incident field inside the dielectric, and the remaining wave travels at the speed c/n ! This proves the Ewald-Oseen extinction theorem.

While deriving the theorem, I solved Maxwell's equations for a single dipole and summed up the fields generated by all the dipoles. I neither used nor defined Maxwell's boundary conditions. I also did not define what a 'medium' is, or what reflected waves and transmitted waves are. These definitions are macroscopic in nature and therefore required in the macroscopic approach. It is exactly these macroscopic definitions that mask the underlying physical phenomena. Yes, they simplify the mathematics a lot but they also lack insight.

6.5 Electric field outside the dielectric

Now I solve equation (34), shown in equation (45) below, for $z < 0$ to find the reflected field. The derivation is very similar to the previous one.

$$E_t(z) = E_{i0}e^{ik_0z} + \frac{1}{2}ik_0(\epsilon_r - 1) \int_0^\infty dz' E_t(z')e^{ik_0|z-z'|} \quad (45)$$

An important note: $E_t(z)$ that I am trying to find is for $z < 0$, but $E_t(z')$ inside the integral is for $z' > 0$ because that is where the integral is evaluated (where the dipoles are located). In other words, $E_t(z')$ is the total field I have found for $z' > 0$ in the previous section. Therefore, using $E_t(z') = E_{t0}e^{ikz'}$,

$$\begin{aligned} E_t(z) &= E_{i0}e^{ik_0z} + \frac{1}{2}ik_0(\epsilon_r - 1) \int_0^\infty dz' E_{t0}e^{ikz'} e^{ik_0(z'-z)} \\ &= E_{i0}e^{ik_0z} + \frac{1}{2}ik_0E_{t0}(\epsilon_r - 1)e^{-ik_0z} \int_0^\infty dz' e^{i(k+k_0)z'} \\ &= E_{i0}e^{ik_0z} + \frac{k_0(\epsilon_r - 1)}{2(k + k_0)} E_{t0}e^{-ik_0z} \\ &= E_{i0}e^{ik_0z} + \frac{k_0(n + 1)(n - 1)}{2k_0(n + 1)} E_{t0}e^{-ik_0z} \end{aligned} \quad (46)$$

Finally substituting for E_{t0} from equation (43),

$$E_t(z) = E_{i0}e^{ik_0z} - \frac{n - 1}{n + 1} E_{i0}e^{-ik_0z} \quad (47)$$

The backward propagating part of the expression (e^{-ik_0z}) is the reflected field that I expected to find.

7 A word on the history

Initially, it was thought that the first layer of dipoles in Figure 4 cancels the incident wave completely, making it “extinct”. Later, realizing that this could not possibly be true, an extinction depth was defined. After the incident wave passes through this depth, it becomes extinct and only the transmitted wave that we observe remains inside the dielectric. However, we currently understand that we should take all the dipoles into account because they act collectively to create the transmitted wave. Introducing a transient step such as the extinction depth or cancellation by the first layer of dipoles only raises more questions: What happens to the transmitted wave as it travels towards infinity? Why doesn’t it change due to the dipoles at further layers? After the incident wave becomes extinct, how can the transmitted wave travel at a lower speed than c in free space without being affected by the following layers of dipoles? The approach adopted in this paper eliminates all those questions.

8 Appendices

8.1 Appendix A

See Appendix C for the definitions of gradient, divergence, curl and convective derivative.

$$\mathbf{A}(\mathbf{r}) = -\frac{i\omega\mu_0}{4\pi}\mathbf{p}\frac{e^{ikR}}{R} \quad (48)$$

$$\begin{aligned} \mathbf{H} &= \frac{1}{\mu_0}\nabla \times \mathbf{A} \quad (49) \\ &= -\frac{i\omega}{4\pi}\nabla \times \left(\mathbf{p}\frac{e^{ikR}}{R}\right) \\ &= -\frac{i\omega}{4\pi}\left[\frac{e^{ikR}}{R}\nabla \times \mathbf{p} + \nabla \left(\frac{e^{ikR}}{R}\right) \times \mathbf{p}\right] \end{aligned}$$

where I used $\nabla \times (\phi\mathbf{A}) = \phi\nabla \times \mathbf{A} + \nabla\phi \times \mathbf{A}$. Notice that $\nabla \times \mathbf{p} = 0$, since \mathbf{p} is a constant with respect to position.

$$\begin{aligned} \nabla \left(\frac{e^{ikR}}{R}\right) &= \frac{1}{R}\nabla e^{ikR} + e^{ikR}\nabla \frac{1}{R} \quad (50) \\ &= ik\frac{e^{ikR}}{R}\hat{\mathbf{R}} - \frac{e^{ikR}}{R^2}\hat{\mathbf{R}} \\ &= e^{ikR}\left(\frac{ik}{R} - \frac{1}{R^2}\right)\hat{\mathbf{R}} \end{aligned}$$

Substituting (50) into (49),

$$\begin{aligned} \mathbf{H} &= -\frac{i\omega}{4\pi}e^{ikR}\left(\frac{ik}{R} - \frac{1}{R^2}\right)\hat{\mathbf{R}} \times \mathbf{p} \quad (51) \\ &= \frac{\omega}{4\pi}e^{ikR}\left(\frac{k}{R} + \frac{i}{R^2}\right)\hat{\mathbf{R}} \times \mathbf{p} \end{aligned}$$

Now, I find the electric field.

$$\begin{aligned} \mathbf{E} &= \frac{i}{\omega\epsilon_0}\nabla \times \mathbf{H} \quad (52) \\ &= \frac{i}{4\pi\epsilon_0}\nabla \times \left[\left(\frac{ke^{ikR}}{R} + \frac{ie^{ikR}}{R^2}\right)\hat{\mathbf{R}} \times \mathbf{p}\right] \\ &= \frac{i}{4\pi\epsilon_0}\left[\left(\frac{ke^{ikR}}{R} + \frac{ie^{ikR}}{R^2}\right)\nabla \times (\hat{\mathbf{R}} \times \mathbf{p}) + \nabla \left(\frac{ke^{ikR}}{R} + \frac{ie^{ikR}}{R^2}\right) \times (\hat{\mathbf{R}} \times \mathbf{p})\right] \end{aligned}$$

Calculating the terms in (52) individually,

$$\nabla \left(\frac{ke^{ikR}}{R} \right) = e^{ikR} \left(\frac{ik^2}{R} - \frac{k}{R^2} \right) \hat{\mathbf{R}} \quad (53)$$

$$\nabla \left(\frac{ie^{ikR}}{R^2} \right) = e^{ikR} \left(\frac{-k}{R^2} - \frac{2i}{R^3} \right) \hat{\mathbf{R}} \quad (54)$$

Also,

$$\begin{aligned} \hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \mathbf{p}) &= (\hat{\mathbf{R}} \cdot \mathbf{p})\hat{\mathbf{R}} - (\hat{\mathbf{R}} \cdot \hat{\mathbf{R}})\mathbf{p} \\ &= \hat{\mathbf{R}}(\hat{\mathbf{R}} \cdot \mathbf{p}) - \mathbf{p} \end{aligned} \quad (55)$$

which I will need when we substitute (53) and (54) into (52). Moving onto the next term in (52),

$$\nabla \times (\hat{\mathbf{R}} \times \mathbf{p}) = \hat{\mathbf{R}}(\nabla \cdot \mathbf{p}) + (\mathbf{p} \cdot \nabla)\hat{\mathbf{R}} - \mathbf{p}(\nabla \cdot \hat{\mathbf{R}}) - (\hat{\mathbf{R}} \cdot \nabla)\mathbf{p} \quad (56)$$

On the right hand side of (56), the first and the last term are zero because, again, \mathbf{p} is a constant with respect to position. The third term is

$$\begin{aligned} \mathbf{p}(\nabla \cdot \hat{\mathbf{R}}) &= \mathbf{p} \left(\nabla \cdot \frac{\mathbf{R}}{R} \right) \\ &= \mathbf{p} \left(\frac{1}{R} \nabla \cdot \mathbf{R} + \mathbf{R} \cdot \nabla \frac{1}{R} \right) \\ &= \mathbf{p} \left(\frac{3}{R} - \mathbf{R} \cdot \frac{\hat{\mathbf{R}}}{R^2} \right) \\ &= \frac{2}{R} \mathbf{p} \end{aligned} \quad (57)$$

The second term is

$$\begin{aligned} (\mathbf{p} \cdot \nabla)\hat{\mathbf{R}} &= p_r \frac{\partial 1}{\partial R} \hat{\mathbf{R}} + \frac{p_\theta}{R} \hat{\boldsymbol{\theta}} + \frac{p_\phi}{R} \hat{\boldsymbol{\phi}} \\ &= \frac{p_\theta \hat{\boldsymbol{\theta}} + p_\phi \hat{\boldsymbol{\phi}}}{R} \end{aligned} \quad (58)$$

Notice that $p_\theta \hat{\boldsymbol{\theta}} + p_\phi \hat{\boldsymbol{\phi}}$ is nothing but \mathbf{p} without its $\hat{\mathbf{R}}$ component, which I can write as $\mathbf{p} - \hat{\mathbf{R}}(\hat{\mathbf{R}} \cdot \mathbf{p})$. So,

$$(\mathbf{p} \cdot \nabla)\hat{\mathbf{R}} = \frac{\mathbf{p} - \hat{\mathbf{R}}(\hat{\mathbf{R}} \cdot \mathbf{p})}{R} \quad (59)$$

Equation (56) finally becomes

$$\begin{aligned}\nabla \times (\hat{\mathbf{R}} \times \mathbf{p}) &= \frac{\mathbf{p} - \hat{\mathbf{R}}(\hat{\mathbf{R}} \cdot \mathbf{p})}{R} - \frac{2}{R}\mathbf{p} \\ &= \frac{-\mathbf{p} - \hat{\mathbf{R}}(\hat{\mathbf{R}} \cdot \mathbf{p})}{R}\end{aligned}\quad (60)$$

Putting (53), (54), (55) and (60) into (52), I get

$$\begin{aligned}\mathbf{E} &= \frac{ie^{ikR}}{4\pi\epsilon_0} \left[-\left(\frac{k}{R^2} + \frac{i}{R^3}\right) (\hat{\mathbf{R}}(\hat{\mathbf{R}} \cdot \mathbf{p}) + \mathbf{p}) + \left(\frac{ik^2}{R} - \frac{2k}{R^2} - \frac{2i}{R^3}\right) (\hat{\mathbf{R}}(\hat{\mathbf{R}} \cdot \mathbf{p}) - \mathbf{p}) \right] \\ &= \frac{e^{ikR}}{4\pi\epsilon_0} \left[\left(3\hat{\mathbf{R}}(\hat{\mathbf{R}} \cdot \mathbf{p}) - \mathbf{p}\right) \left(\frac{1}{R^3} - \frac{ik}{R^2}\right) - \left(\hat{\mathbf{R}}(\hat{\mathbf{R}} \cdot \mathbf{p}) - \mathbf{p}\right) \frac{k^2}{R} \right]\end{aligned}\quad (61)$$

8.2 Appendix B

I will evaluate the integrals with respect to dx' and dy' in equation (62).

$$\begin{aligned}E_d(z) &= \frac{\epsilon_r - 1}{4\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_0^{\infty} dz' \\ &\quad \times E_t(z') e^{ik_0 R} \left[\left(3(\hat{\mathbf{R}} \cdot \hat{\mathbf{x}})^2 - 1\right) \left(\frac{1}{R^3} - \frac{ik_0}{R^2}\right) - \left(3(\hat{\mathbf{R}} \cdot \hat{\mathbf{x}})^2 - 1\right) \frac{k_0^2}{R} \right]\end{aligned}\quad (62)$$

Consider Figure 8, which I will use to switch from cartesian to cylindrical coordinates. Notice how I shifted the x and y (or simply ρ) coordinates to the location of the observation point. I can easily do this because the problem is independent of x and y coordinates: The media is homogeneous in x and y directions, the incident plane wave is normal to the interface, and the integrals spans the entire x - y plane. If any of them were not true, then I would need to take into account any effect on the electric field that might occur due to a shift in the coordinates.

Next, I observe from Figure 8 that $\hat{\mathbf{R}} \cdot \hat{\mathbf{x}} = -\rho \cos \phi / R$. Also, $R^2 = \rho^2 + (z - z')^2$ and $RdR = \rho d\rho$. Now I can replace the integrals in equation (62) as

$$\begin{aligned}E_d(z) &= \frac{\epsilon_r - 1}{4\pi} \int_0^{\infty} \rho d\rho \int_0^{2\pi} d\phi \int_0^{\infty} dz' \\ &\quad \times E_t(z') e^{ik_0 R} \left[\left(\frac{3\rho^2 \cos^2 \phi}{R^2} - 1\right) \left(\frac{1}{R^3} - \frac{ik_0}{R^2}\right) - \left(\frac{\rho^2 \cos^2 \phi}{R^2} - 1\right) \frac{k_0^2}{R} \right]\end{aligned}\quad (63)$$

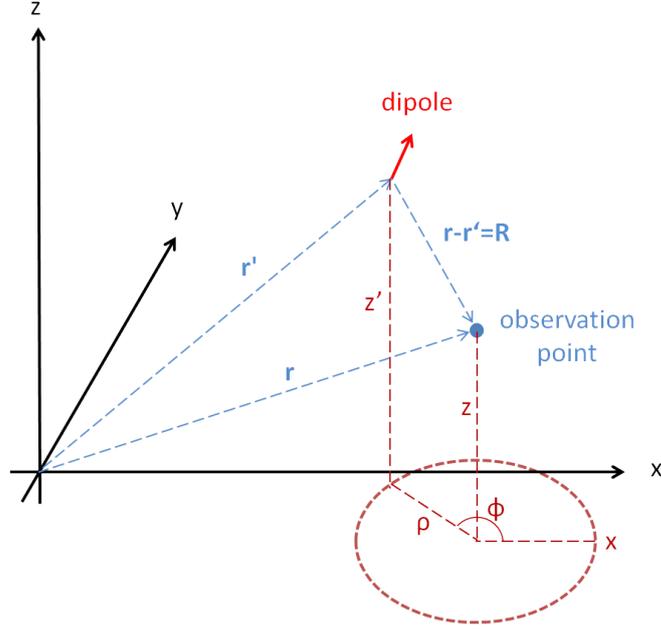


Figure 8: Coordinate switch from cartesian to cylindrical

Evaluating the integral with respect to ϕ ,

$$E_d(z) = \frac{\epsilon_r - 1}{4} \int_0^\infty \rho d\rho \int_0^\infty dz' \times E_t(z') e^{ik_0 R} \left[\left(\frac{3\rho^2}{R^2} - 2 \right) \left(\frac{1}{R^3} - \frac{ik_0}{R^2} \right) - \left(\frac{\rho^2}{R^2} - 2 \right) \frac{k_0^2}{R} \right] \quad (64)$$

Replacing ρ ,

$$\begin{aligned} E_d(z) &= \frac{\epsilon_r - 1}{4} \int_{|z-z'|}^\infty R dR \int_0^\infty dz' \\ &\times E_t(z') e^{ik_0 R} \left[\left(1 - \frac{3(z-z')^2}{R^2} \right) \left(\frac{1}{R^3} - \frac{ik_0}{R^2} \right) + \left(1 + \frac{(z-z')^2}{R^2} \right) \frac{k_0^2}{R} \right] \\ &= \frac{\epsilon_r - 1}{4} \int_{|z-z'|}^\infty dR \int_0^\infty dz' \\ &\times E_t(z') e^{ik_0 R} \left[\frac{1}{R^2} - \frac{ik_0}{R} - \frac{3(z-z')^2}{R^4} + \frac{3ik_0(z-z')^2}{R^3} + k_0^2 + \frac{k_0^2(z-z')^2}{R^2} \right] \end{aligned} \quad (65)$$

The limits of $d\rho$ integral is from 0 to infinity, but $R = |z - z'|$ when $\rho = 0$. So, the limits of dR integral is from $|z - z'|$ to infinity. To evaluate dR integral, consider

the following integral which represents all the terms in (65).

$$F_n = \int_{|z-z'|}^{\infty} dR \frac{e^{ik_0 R}}{R^n}, \quad n = 1, 2, 3, \dots \quad (66)$$

I will calculate a few of the first terms and see if I can find all the terms I need to calculate the integral in (65).

$$F_0 = \int_{|z-z'|}^{\infty} dR e^{ik_0 R} = \frac{e^{ik_0 R}}{ik_0} \Big|_{|z-z'|}^{\infty} = -\frac{1}{ik_0} e^{ik_0 |z-z'|} \quad (67)$$

The reason why I am able to set the upper limit to zero is given in the main text.

$$\begin{aligned} F_1 &= \int_{|z-z'|}^{\infty} dR \frac{e^{ik_0 R}}{R} \\ &= \frac{e^{ik_0 R}}{ik_0 R} \Big|_{|z-z'|}^{\infty} + \int_{|z-z'|}^{\infty} dR \frac{e^{ik_0 R}}{ik_0 R^2} \\ &= \frac{1}{ik_0} \left(-\frac{e^{ik_0 |z-z'|}}{|z-z'|} + \int_{|z-z'|}^{\infty} dR \frac{e^{ik_0 R}}{ik_0 R^2} \right) \\ &= \frac{1}{ik_0} \left(F_2 - \frac{e^{ik_0 |z-z'|}}{|z-z'|} \right) \end{aligned} \quad (68)$$

$$\begin{aligned} F_2 &= \int_{|z-z'|}^{\infty} dR \frac{e^{ik_0 R}}{R^2} \\ &= \frac{e^{ik_0 R}}{ik_0 R^2} \Big|_{|z-z'|}^{\infty} + \int_{|z-z'|}^{\infty} dR \frac{2e^{ik_0 R}}{ik_0 R^3} \\ &= \frac{1}{ik_0} \left(2F_3 - \frac{e^{ik_0 |z-z'|}}{|z-z'|^2} \right) \end{aligned} \quad (69)$$

$$\begin{aligned} F_3 &= \int_{|z-z'|}^{\infty} dR \frac{e^{ik_0 R}}{R^3} \\ &= \frac{e^{ik_0 R}}{ik_0 R^3} \Big|_{|z-z'|}^{\infty} + \int_{|z-z'|}^{\infty} dR \frac{3e^{ik_0 R}}{ik_0 R^4} \\ &= \frac{1}{ik_0} \left(3F_4 - \frac{e^{ik_0 |z-z'|}}{|z-z'|^3} \right) \end{aligned} \quad (70)$$

I don't seem to be able to get any closed form expressions except for F_0 (By the way, I used integration by parts). However, I see a recursion relation!

$$F_n = \frac{1}{ik_0} \left(nF_{n+1} - \frac{e^{ik_0|z-z'|}}{|z-z'|^n} \right) \quad (71)$$

All I can do now is use this relation in (65) and hope for the best. Specifically, "the best" means that all the terms I cannot explicitly calculate in (65) will hopefully cancel each other out. Let's find out:

$$E_d(z) = \frac{\epsilon_r - 1}{4} \int_0^\infty dz' E_t(z') \\ \times [F_2 - ik_0 F_1 - 3(z-z')^2 F_4 + 3ik_0(z-z')^2 F_3 + k_0^2 F_0 + k_0^2(z-z')^2 F_2]$$

$$E_d(z) = \frac{\epsilon_r - 1}{4} \int_0^\infty dz' E_t(z') \\ \times \left[F_2 - F_2 + \frac{e^{ik_0|z-z'|}}{|z-z'|} - 3(z-z')^2 F_4 + 3ik_0(z-z')^2 \left(3F_4 - \frac{e^{ik_0|z-z'|}}{|z-z'|^3} \right) \right. \\ \left. + k_0^2 F_0 - ik_0(z-z')^2 \left(2F_3 - \frac{e^{ik_0|z-z'|}}{|z-z'|^2} \right) \right]$$

$$E_d(z) = \frac{\epsilon_r - 1}{4} \int_0^\infty dz' E_t(z') \\ \times \left[-2 \frac{e^{ik_0|z-z'|}}{|z-z'|} - 6(z-z')^2 F_4 \right. \\ \left. + k_0^2 F_0 + ik_0 e^{ik_0|z-z'|} - (z-z')^2 \left(6F_4 - \frac{2e^{ik_0|z-z'|}}{|z-z'|^3} \right) \right]$$

and finally,

$$E_d(z) = \frac{ik_0(\epsilon_r - 1)}{2} \int_0^\infty dz' E_t(z') e^{ik_0|z-z'|} \quad (72)$$

8.3 Appendix C

Note $f = f(r, \theta, \phi)$, $\mathbf{A} = A_r \hat{\mathbf{r}} + A_\theta \hat{\boldsymbol{\theta}} + A_\phi \hat{\boldsymbol{\phi}}$ and $\mathbf{B} = B_r \hat{\mathbf{r}} + B_\theta \hat{\boldsymbol{\theta}} + B_\phi \hat{\boldsymbol{\phi}}$ in spherical coordinates.

Draw the coordinates and axes in a graph!!!

Gradient:

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} \quad (73)$$

Divergence:

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\sin \theta A_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \quad (74)$$

Curl:

$$\begin{aligned} \nabla \times \mathbf{A} = & \frac{1}{r \sin \theta} \left[\frac{\partial(\sin \theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right] \hat{\mathbf{r}} \\ & + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial(r A_\phi)}{\partial r} \right] \hat{\boldsymbol{\theta}} \\ & + \frac{1}{r} \left[\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}} \end{aligned} \quad (75)$$

Convective derivative:

$$\begin{aligned} (\mathbf{A} \cdot \nabla) \mathbf{B} = & \left(A_r \frac{\partial B_r}{\partial r} + \frac{A_\theta}{r} \frac{\partial B_r}{\partial \theta} + \frac{A_\phi}{r \sin \theta} \frac{\partial B_r}{\partial \phi} - \frac{A_\theta B_\theta + A_\phi B_\phi}{r} \right) \hat{\mathbf{r}} \\ & + \left(A_r \frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{A_\phi}{r \sin \theta} \frac{\partial B_\theta}{\partial \phi} + \frac{A_\theta B_r}{r} - \frac{A_\phi B_\phi \cot \theta}{r} \right) \hat{\boldsymbol{\theta}} \\ & + \left(A_r \frac{\partial B_\phi}{\partial r} + \frac{A_\theta}{r} \frac{\partial B_\phi}{\partial \theta} + \frac{A_\phi}{r \sin \theta} \frac{\partial B_\phi}{\partial \phi} + \frac{A_\phi B_r}{r} + \frac{A_\phi B_\theta \cot \theta}{r} \right) \hat{\boldsymbol{\phi}} \end{aligned} \quad (76)$$

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